



Quantum game theory. I

David A. Meyer

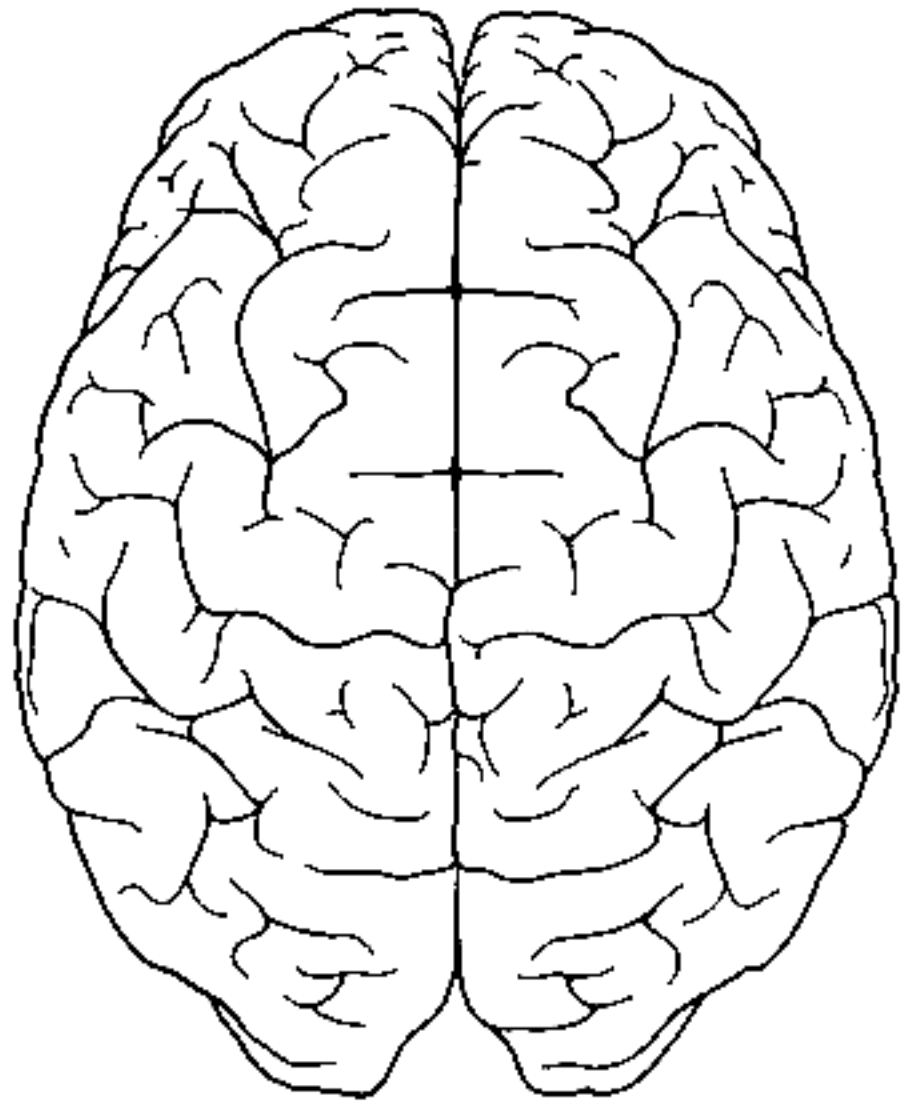
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Fourth Canadian Summer School on Quantum Information
Institute for Quantum Computing & University of Waterloo

21 June 2004

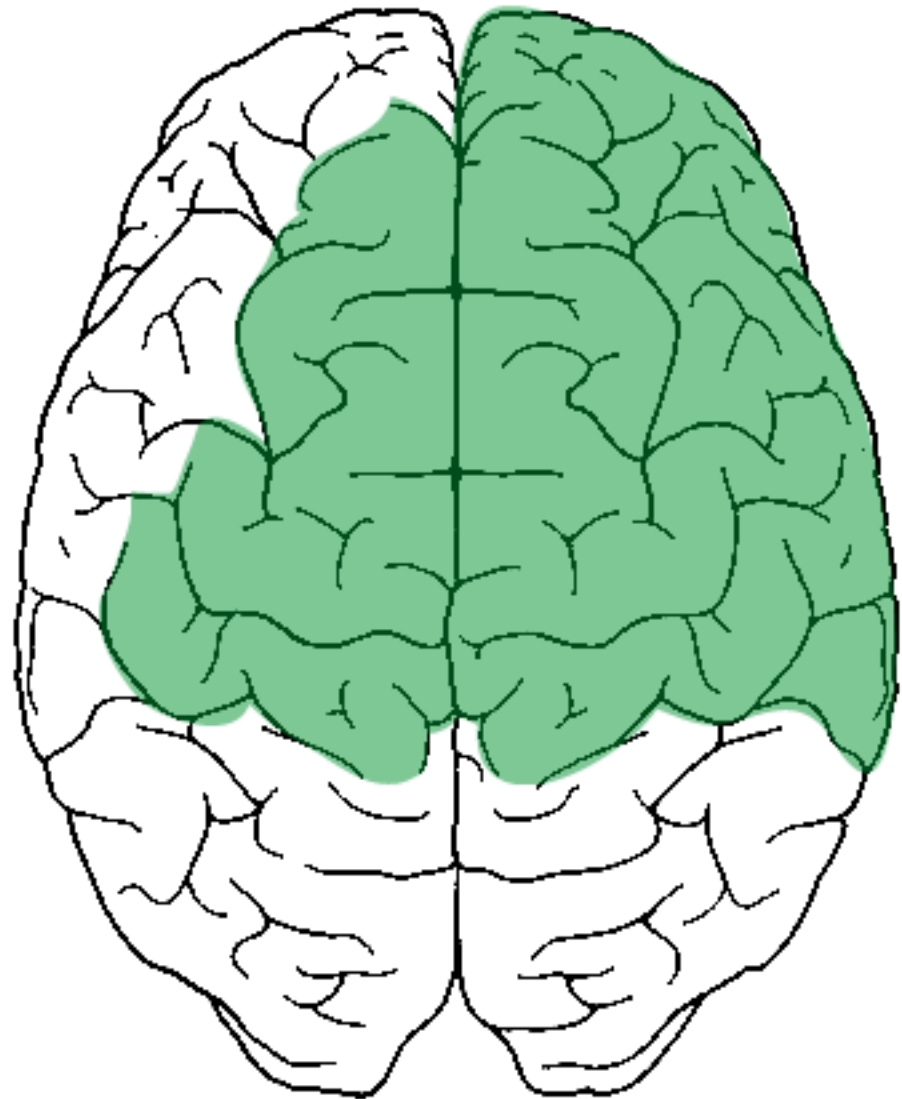


von Neumann



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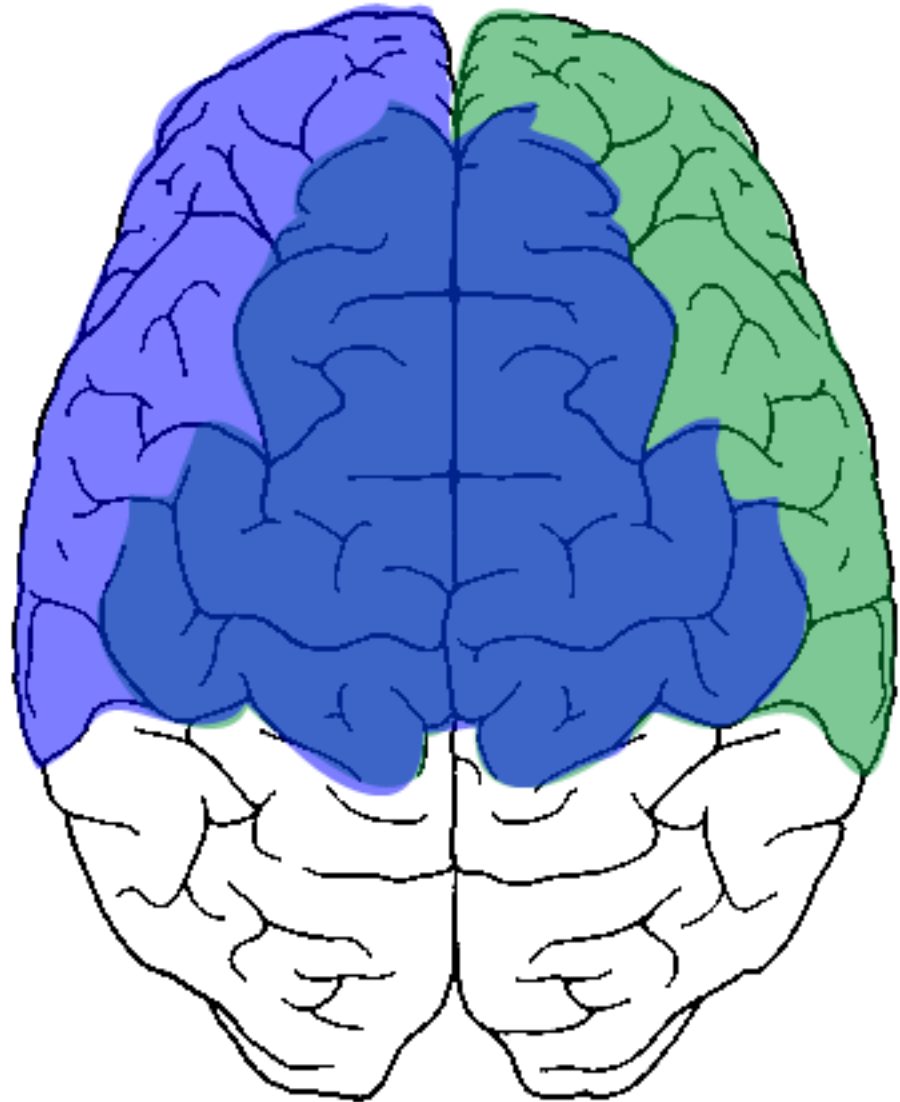
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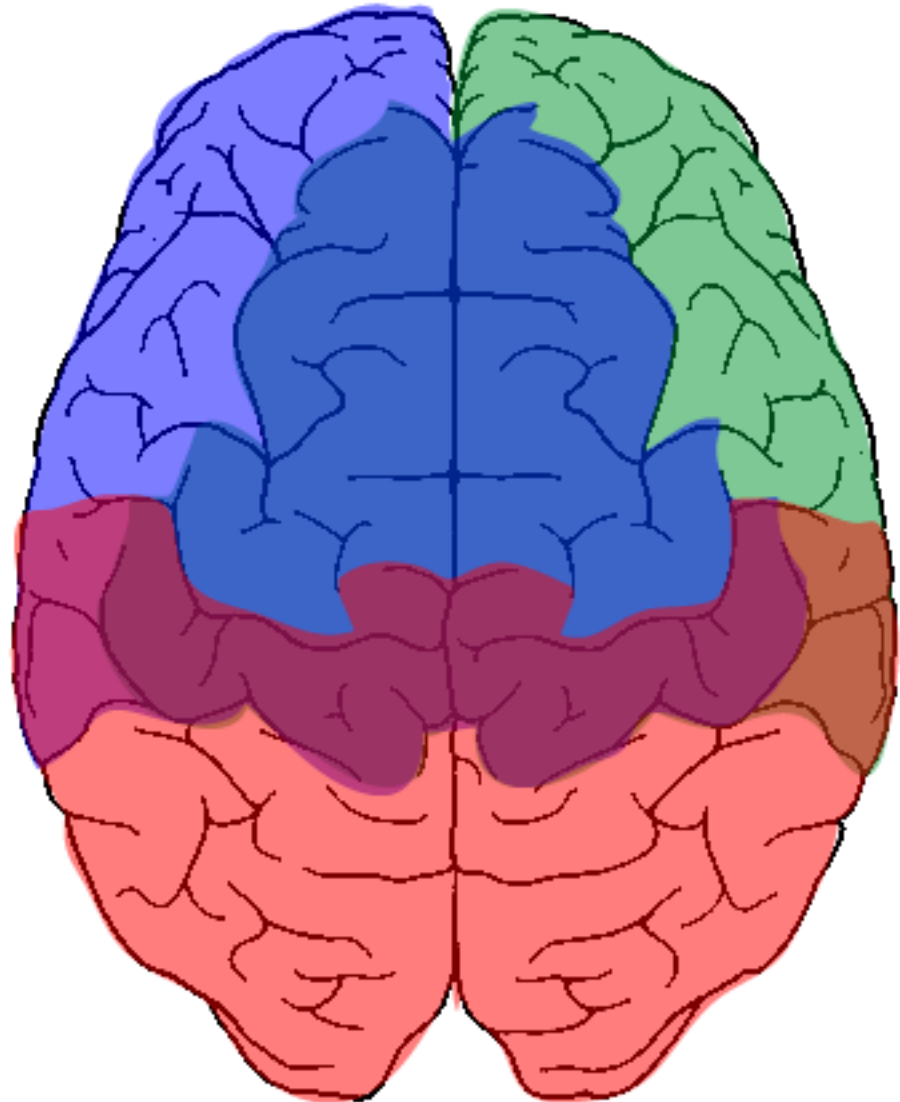


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So quantum games are within experimental/technological reach.

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... and how to fix it.

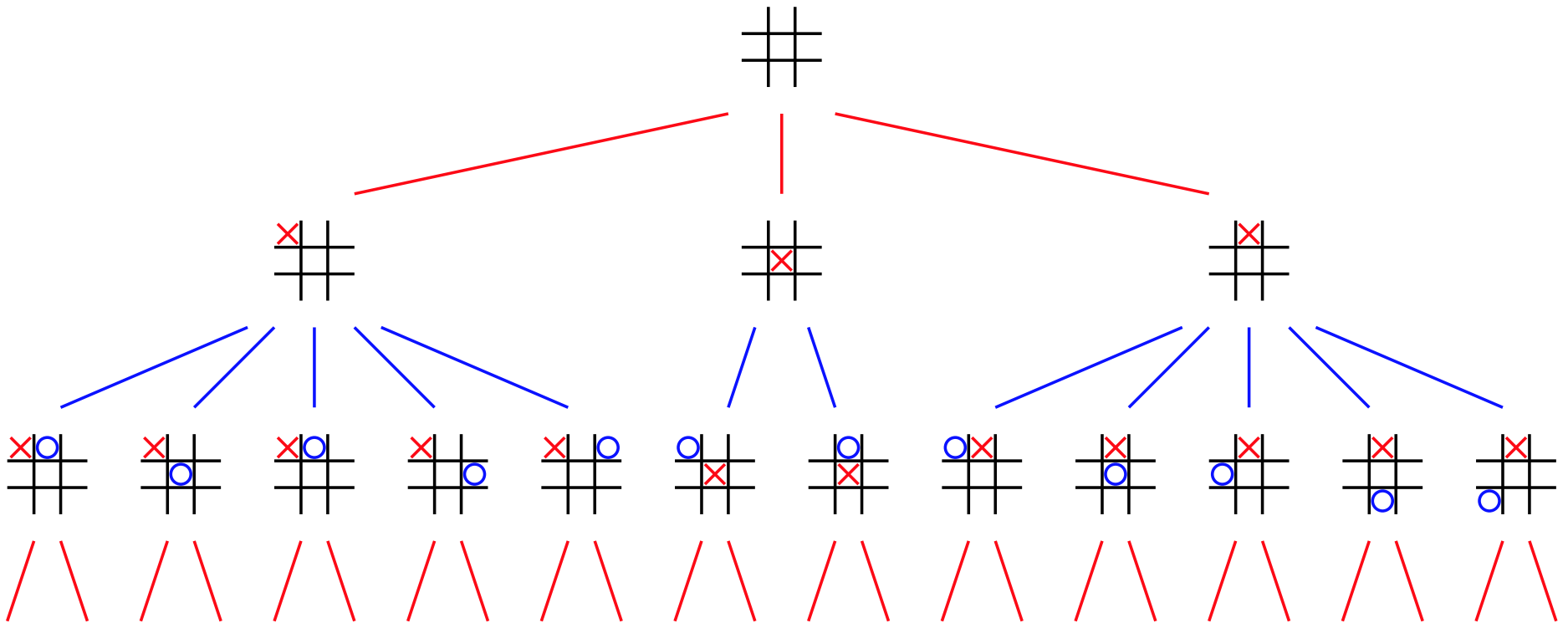
Classical game theory

is a formalism for analyzing strategic interactions.

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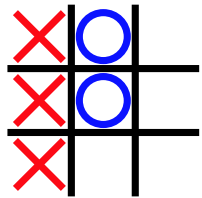
is a formalism for analyzing strategic interactions.

EXAMPLE: The possible sequences of moves in **Tic-Tac-Toe** are paths in a **game tree**:

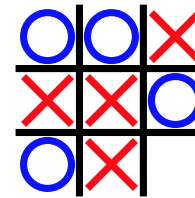


Extensive form

Paths terminate at configurations for which the outcome is defined, e.g.:



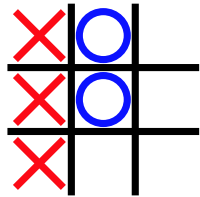
and



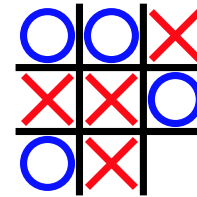
which have **payoffs** $(1, -1)$ and $(0, 0)$, respectively.

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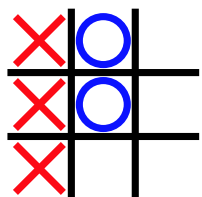


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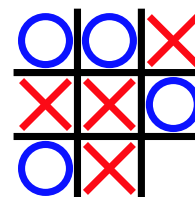
(When the sum of the payoffs at each outcome is 0, the game is called **zero-sum**.)

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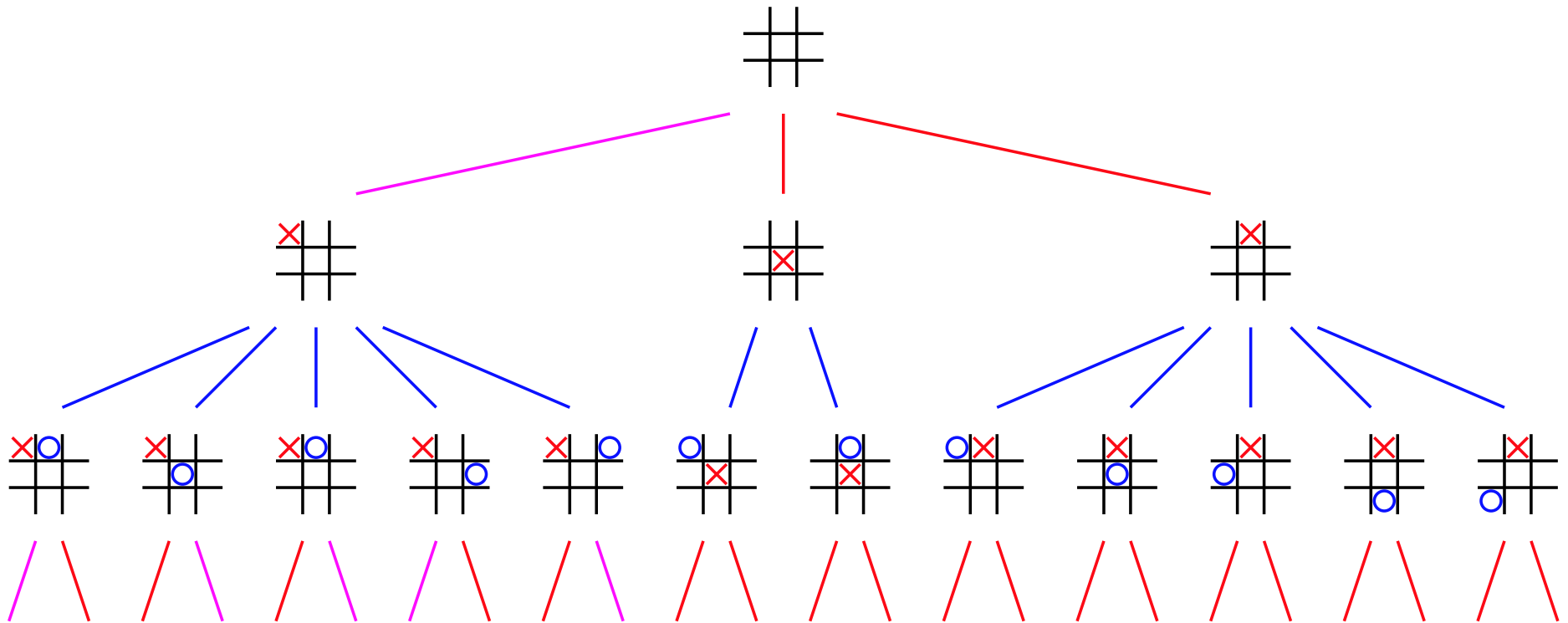
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This is the **extensive form** description of the game, what we might call **dynamic**.

Strategies

A (pure) strategy for a player is a specification of how to play in each possible situation.

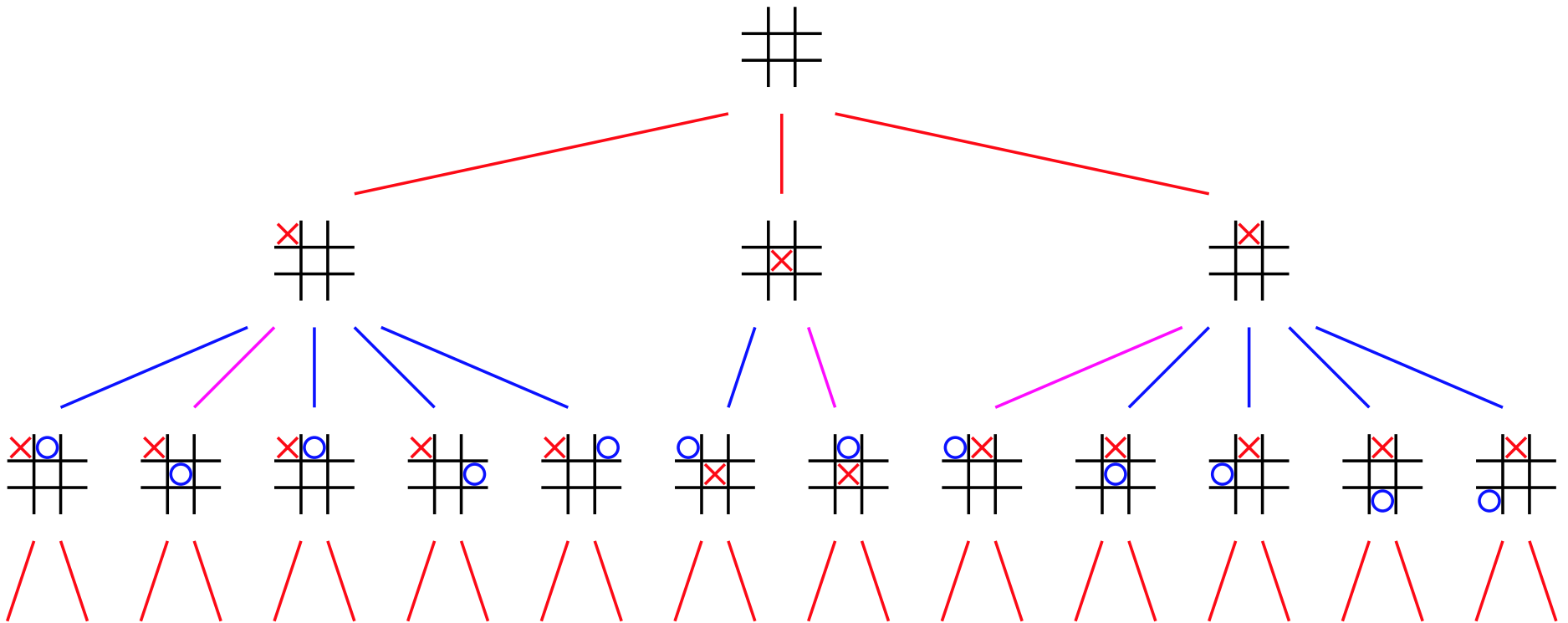
For player 1 in our example:



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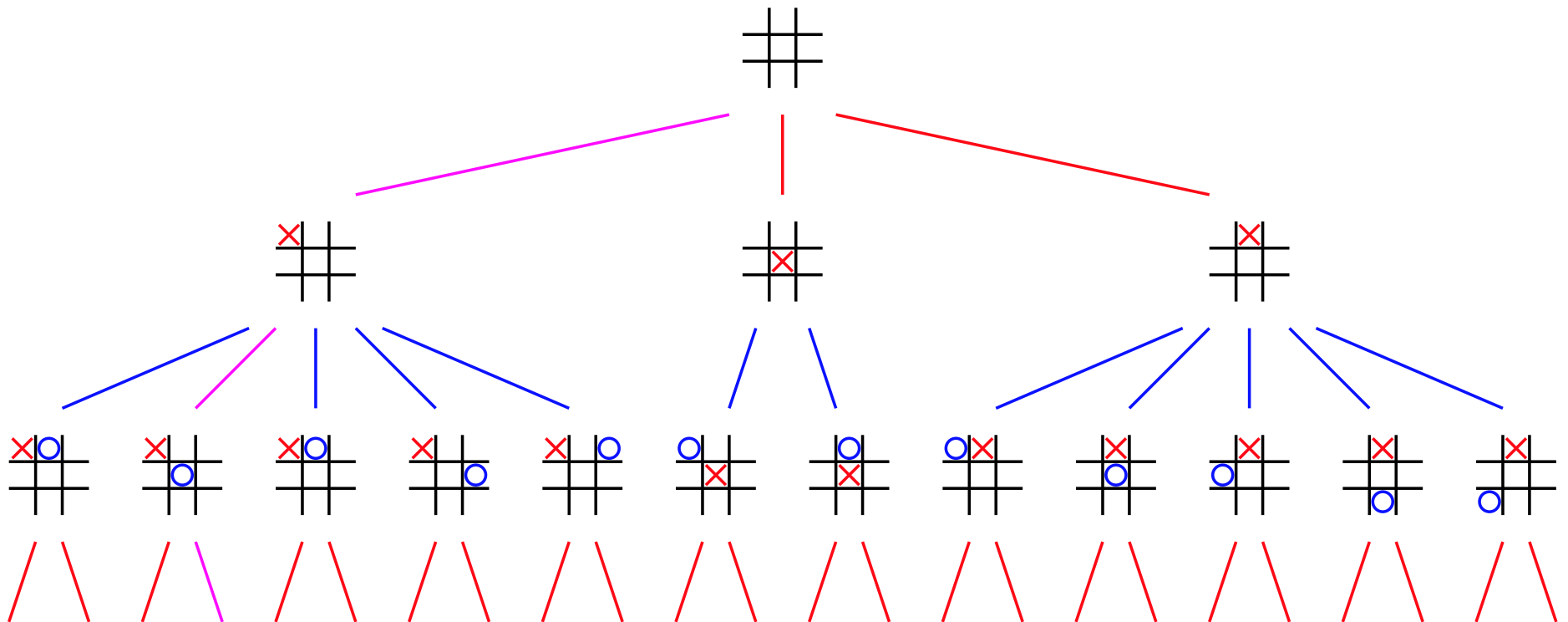
and for player 2:



Strategies

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So a pair of strategies, one for each player, specifies a path through the game tree:



and hence a pair of payoffs.

Strategic form

The **strategic form** description of a game (what we might call **static**) consists of a **payoff matrix** with rows indexed by player **1**'s strategies (S_1) and columns indexed by player **2**'s strategies (S_2).

The $s = (s_1, s_2)$ element of the payoff matrix, $u_s = u_{s_1 s_2}$, is the ordered pair of payoffs that is determined by the corresponding path through the game tree.

(All these definitions generalize to more than 2 players.)

Nash equilibria

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EXAMPLE: Let $S_1 = \{C, D\} = S_2$ and consider the game with payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \left(\begin{array}{cc} (5, 5) & (2, 4) \\ (4, 2) & (1, 1) \end{array} \right) \end{array}$$

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	C	D
C	$(5, 5)$	$(2, 4)$
D	$(4, 2)$	$(1, 1)$

(C, C) is a Nash equilibrium.

Nash equilibria

EXAMPLE: Rock/Paper/Scissors has payoff matrix:

$$\begin{array}{c} R \\ P \\ S \end{array} \begin{array}{ccc} R & P & S \\ \left(\begin{array}{ccc} (0, 0) & (-1, 1) & (1, -1) \\ (1, -1) & (0, 0) & (-1, 1) \\ (-1, 1) & (1, -1) & (0, 0) \end{array} \right) \end{array}$$

This game has no pure strategy Nash equilibrium.

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A set of mixed strategies, one for each player, is a **(Nash) equilibrium** if no player can improve his expected payoff,

$$E[u^{(i)}] = \sum_{s \in S} u_s^{(i)} p_{s_1} p_{s_2},$$

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THEOREM (von Neumann): Every zero-sum game has a (Nash) equilibrium in the space of mixed strategies. While there may be multiple equilibria, the expected payoffs are unique.

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... at the cost of sometimes having multiple equilibria, with different expected payoffs.

Multiple equilibria

EXAMPLE: The **Battle of the Sexes** game (Luce and Raiffa, 1957) has payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} F & S \end{array} \\ \begin{array}{c} F \\ S \end{array} & \left(\begin{array}{cc} (3, 1) & (0, 0) \\ (0, 0) & (1, 3) \end{array} \right) \end{array}$$

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So is $(\frac{3}{4}F + \frac{1}{4}S, \frac{1}{4}F + \frac{3}{4}S)$, which has expected payoffs $(\frac{3}{4}, \frac{3}{4})$, also different.

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We would like to fix this, *i.e.*, get players to Pareto efficient outcomes.

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But this doesn't help: if the state of the game is

$$\sum_{s_1 \in S_1} \alpha_{s_1} |s_1\rangle \otimes \sum_{s_2 \in S_2} \beta_{s_2} |s_2\rangle,$$

since $|s_1 s_2\rangle$ is the measurement outcome with probability $|\alpha_{s_1} \beta_{s_2}|^2 = |\alpha_{s_1}|^2 |\beta_{s_2}|^2$, we can simulate this outcome with a **mixed** strategy pair:

$$\begin{aligned} p_{s_1} &= |\alpha_{s_1}|^2 & \forall s_1 \in S_1 \\ p_{s_2} &= |\beta_{s_2}|^2 & \forall s_2 \in S_2. \end{aligned}$$

Quantum strategies

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The hope is that entanglement buys you something, and “more” entanglement buys more.

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ELW: In order to guarantee that this protocol generalizes the classical game, we identify I with C and X with D , and choose J such that if both players play I or X , the payoffs are as in the classical game. *I.e.*,

$$[J, I \otimes X] = 0 = [J, X \otimes I] = [J, X \otimes X].$$

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CLAIM (EWL): (X, X) is no longer a Nash equilibrium, but there is a (symmetric) Nash equilibrium with expected payoffs $(5, 5)$.

THEOREM (Benjamin and Hayden): There are no Nash equilibria among pairs of unitary transformations.

Implications

We should really consider **general quantum operations**; by standard game-theoretic theorems about convex, compact sets, there will be (usually many) Nash equilibria. Some may be better than the classical equilibria.

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Tomorrow: So why do **van Enk** and **Pike** say this is no fun?