Foundations and Interpretation of Quantum Theory,
AMATH 900/ AMATH 495/ Phys 490
Problem Sheet 1
Due: February 11th

1. *Pure vs Mixed States and their Matrix Representations.*

   Consider the pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, which is a coherent superposition of the basis states $\{|0\rangle, |1\rangle\}$ for a two-level system.

   a) Express the state operator $\hat{\rho}_p = |\psi\rangle\langle\psi|$ as a matrix in the $\{|0\rangle, |1\rangle\}$ basis.
   b) Express the state operator $\hat{\rho}_m = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$ as a matrix in the same basis.
   c) Consider the new basis $\{|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\}$. Express both $\hat{\rho}_p$ and $\hat{\rho}_m$ as matrices in this new basis.
   d) Determine a condition on α, β so that $\hat{\rho}_m$ is a diagonal matrix in every basis.
   e) What is the relationship between the coherence/incoherence of a superposition over two basis vectors and the off-diagonal matrix elements of the state operator expressed in the basis defined by these vectors?

2. *Schmidt Decomposition and Degree of Purity.*

   The Schmidt decomposition is defined as follows: for all $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there are bases $\{|\chi_{A,i}\rangle\}$ and $\{|\chi_{B,i}\rangle\}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively such that

   $$|\psi\rangle = \sum_i \alpha_i |\chi_{A,i}\rangle \otimes |\chi_{B,i}\rangle \tag{1}$$

   where $\alpha_i \geq 0$, $\sum_i \alpha_i^2 = 1$. Note that a state $|\psi\rangle$ is 'entangled' if and only if the number of non-zero α_i is greater than 1.

   Use the Schmidt decomposition to show that

   $$\mathrm{Tr}(\hat{\rho}_A^2) = \mathrm{Tr}(\hat{\rho}_B^2) \qquad \text{if} \qquad |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B,$$

   where the reduced density operators are defined by partial traces: $\rho_A = \mathrm{Tr}_B(\rho)$ and $\rho_A = \mathrm{Tr}_B(\rho)$.

3. *Coherent and incoherent superpositions.*

   If we have two orthonormal states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, we can 'add' them to give another normalized state ρ in at least two different ways. We can add them incoherently,

   $$\rho_{\text{incoh}} = p_1|\Psi_1\rangle\langle\Psi_1| + p_2|\Psi_2\rangle\langle\Psi_2|,$$

   with probabilities p_1 and p_2, or we can add them coherently,

   $$\Psi = \sqrt{p_1}|\Psi_1\rangle + e^{i\theta}\sqrt{p_2}|\Psi_2\rangle$$

   defining the state operator,

   $$\rho_{\text{coh}} = |\Psi\rangle\langle\Psi|,$$

   where $\sqrt{p_1}$ and $\sqrt{p_2}$ are amplitudes and $e^{i\theta}$ defines a relative phase.

   a) Calculate $\mathrm{Tr}(\rho_{\text{incoh}}^2)$ and $\mathrm{Tr}(\rho_{\text{coh}}^2)$ and explain how this quantity provides a quantitative measure of the difference between these two states.

b) For a Hilbert space of dimension 2 with basis states $\{|0\rangle, |1\rangle\}$, and with $p_0 = 1/3$ and $p_2 = 2/3$, find an observable (i.e., a 2×2 Hermitian matrix defined in this basis) whose expectation value can distinguish between a coherent and incoherent superposition and find an observable whose expectation value can not distinguish between coherent and incoherent superpositions.

c) Show that the ('Bell') state

$$|\Psi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} = (|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b)/\sqrt{2}$$

(defined for two two-level systems labeled by 'a' and 'b') can not be written as a state operator of the form,

$$\rho = \sum_i p_i \rho_a^i \otimes \rho_b^i,$$

where $p_i > 0$ and therefore explain why you can deduce that $|\Psi_+\rangle$ is entangled.

4. How many independent parameters are required to describe the most general unitary transformation of a two state system? How many independent parameters describe the most general completely positive map?

5. Consider an initial factorable state of two spin-1/2 particles with basis states $|1\rangle$ and $|-1\rangle$ defined as the eigenstates of the Pauli operator σ_z. Let the initial state $|1\rangle\,|1\rangle$ evolve under a unitary operator U, such that $|\Psi\rangle = U\,|1\rangle\,|1\rangle$, where U represents the following sequence of transformations:

$$U = \left[U^{(A)}(\varphi_A) \otimes U^{(B)}(\varphi_B)\right] U_{AB}(J) \left[U^{(A)}(\theta_A) \otimes U^{(B)}(\theta_B)\right]$$

where

$$U^{(A,B)}(\varphi_{A,B}) = e^{-i\varphi_{A,B}\sigma_x^{(A,B)}}$$

and,

$$U_{AB}(J) = e^{-i\frac{J}{4}\sigma_Z^{(A)}\otimes\sigma_Z^{(B)}}$$

and let $\theta_A = \theta_B = \pi/4$.

1) Calculate the final state reduced density operator for system A, ie, $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$.

2) For which values of J is system A "decohered" by its interaction with system B?

3) Find the value of J such that $\mathrm{Tr}(\rho_A^2) = 1/2$.

4) Show that $\mathrm{Tr}(\rho_A^2)$ does not depend on φ_A or φ_B. Explain the physical significance of this.

5) Show that ρ_A can not depend on φ_B. Explain the physical significance of this.