

Introduction to Quantum Information Processing
(CS667, C&O681, PHYS767, CS467, C&O481, PHYS467)
Fall, 2004

Assignment 1

Due date: October 15, 2004

Each of the seven questions is worth 5 marks.

1. **Distinguishing between quantum states.** Suppose that one of two known quantum states is randomly chosen (according to the uniform distribution), but you don't know which one is selected. Your task is to determine which of the two states it is, with as small an error probability as possible. Your distinguishing procedure can apply a unitary operation (of your choosing) to the qubit and then measure it (in the standard basis).

For example, the pair of states $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ can be perfectly distinguished by applying a Hadamard transform before measuring.

In each case below, describe a distinguishing procedure with as small an error probability as possible.

(a) $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ and $\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

(b) $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ and $\frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ (where $i = \sqrt{-1}$)

(c) $|0\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

(d) Here there are four states: $|0\rangle$, $|1\rangle$, $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, and $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ (uniformly selected).

2. **Simulating a CNOT gate by Hamiltonian evolution.** Consider the Hamiltonian $H = Z \otimes X - Z \otimes I - I \otimes X$, which acts on two qubits ($X = \sigma_x$ and $Z = \sigma_z$ denote the Pauli matrices and I denotes the 2×2 identity matrix). Suppose that one has a physical system enabling one to evolve H for any time interval, and the goal is to simulate a CNOT gate. Find a time t such that e^{-iHt} is a CNOT gate up to a global phase. (Hint: it may help to find a coordinate system in which CNOT is diagonal.)

3. **Entangled states and product states.** For each two-qubit state below, either express it as a product of two one-qubit states, or show that such a factorization is impossible (in the latter case, the states are *entangled*).

(a) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

(b) $\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle$

(c) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

4. **Density matrices.** Recall that density matrices are an alternative formalism for describing quantum states. Every (pure) quantum state can be expressed as a density matrix. Moreover, even states that are probabilistic mixtures of pure states can be expressed as density matrices. Finally two states (pure or mixed) are distinguishable if and only if their density matrices are different.

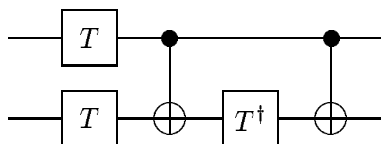
- (a) What is the density matrix of $|0\rangle$? What is the density matrix of the mixed state that is $|0\rangle$ with probability $1/2$ and $|1\rangle$ with probability $1/2$?
- (b) Consider the state that Bob possesses in the teleportation protocol *just before* he receives the two classical bits from Alice. If the state that Alice is teleporting is $\alpha|0\rangle + \beta|1\rangle$ then Bob's state is: $\alpha|0\rangle + \beta|1\rangle$ with probability $1/4$; $\alpha|1\rangle + \beta|0\rangle$ with probability $1/4$; $\alpha|0\rangle - \beta|1\rangle$ with probability $1/4$; $\alpha|1\rangle - \beta|0\rangle$ with probability $1/4$. What is the density matrix of this state?
- (c) Explain why, when Bob has the mixed state in part (b), he has no actual information about the state $\alpha|0\rangle + \beta|1\rangle$.

5. **Tripartite entanglement.** Consider the three-qubit state $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$.

- (a) Can the state be expressed as a product of three one-qubit states? (Justify your answer.)
- (b) Suppose that Alice, Bob, and Carol each possess one of the three qubits of the above state. Now suppose that Carol leaves the scene, taking her qubit with her and ceases all communication with Alice and Bob. Is the two-qubit state that Alice and Bob are left with equivalent to $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$? (Justify your answer.)

6. **On quantum circuits.**

- (a) Show that the unitary transformation corresponding to the circuit



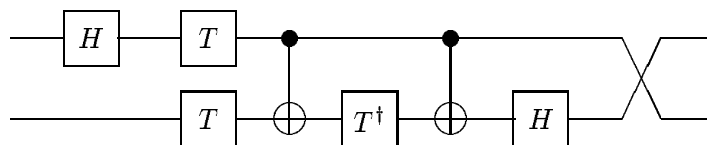
is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

Here the operation T is defined as

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}.$$

- (b) In view of part (a), what is the matrix corresponding to the following circuit?



(We'll see later that this is an important transformation in quantum computing.)

7. **Conversions between Bell states.** Give a 2×2 unitary matrix U such that, if this U is applied to each qubit of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ then the resulting state is $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$. (That is, $(U \otimes U)(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle) = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$ should hold.)