Introduction to Quantum Information Processing

(CS667, CO681, PHYS767, CS467, CO481, PHYS467)
Fall, 2004

Midterm Solutions: Problems 1, 4 (M. Silva), 2, 3 (C. Perez), 5 (D. Cheung).

1. (a) $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

   (b) The Hadamard gate in Dirac notation is

$$H = \left|+\right\rangle \left\langle 0\right| + \left|-\right\rangle \left\langle 1\right| = \frac{1}{\sqrt{2}} \left(\left|0\right\rangle + \left|1\right\rangle\right) \left\langle 0\right| + \frac{1}{\sqrt{2}} \left(\left|0\right\rangle - \left|1\right\rangle\right) \left\langle 1\right|$$

   (c) The eigenvalues of the Hadamard matrix can be inferred directly from the fact that $H^2 = \mathbb{1}$. Thus, the eigenvalues are $\lambda_\pm = \pm 1$. The corresponding eigenvectors can be inferred in the usual way by using $(H - \lambda_\pm \mathbb{1}) \left|H_\pm\right\rangle = 0$ and solving for $\left|H_\pm\right\rangle$, the eigenvectors. That is,

$$\begin{aligned} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - (\pm 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \mp \sqrt{2} & 1 \\ 1 & -1 - \mp\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \end{aligned}$$

   and so, ignoring the $\frac{1}{\sqrt{2}}$ factor, we can say that $v_2 = \frac{v_1}{1 \pm \sqrt{2}}$. If we apply the normalization condition, we find that $|v_1|^2 = \frac{(1 \pm \sqrt{2})^2}{(1 \pm \sqrt{2})^2 + 1}$ and $|v_2|^2 = \frac{1}{(1 \pm \sqrt{2})^2 + 1}$, which yield

$$\left|H_+\right\rangle = \frac{1}{\sqrt{(1+\sqrt{2})^2 + 1}} \Big((1+\sqrt{2}) \left|0\right\rangle + \left|1\right\rangle \Big) \approx 0.9239 \left|1\right\rangle + 0.3827 \left|1\right\rangle$$

   for $\lambda_+ = 1$ and

$$\left|H_-\right\rangle = \frac{1}{\sqrt{(1-\sqrt{2})^2 + 1}} \Big((1-\sqrt{2}) \left|0\right\rangle + \left|1\right\rangle \Big) \approx -0.3827 \left|0\right\rangle + 0.9239 \left|1\right\rangle$$

   for $\lambda_- = -1$.

   An alternative approach is to recall the description of single qubit unitaries as rotations of the Bloch sphere along some axis $\hat{n}$. Since $H = \frac{1}{\sqrt{2}} (X + Z)$, it is clear that $\hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$ and thus the eigenvectors must lie on the surface of the Bloch sphere along that axis. This immediatelly yields the eigenvector (corresponding to eignevalue 1)

$$\left|H_+\right\rangle = \cos\frac{\pi}{8} \left|0\right\rangle + \sin\frac{\pi}{8} \left|1\right\rangle ,$$

   from the geometrical parameterization of a pure state in the direction $\hat{n}$ in the Bloch sphere by the angles $\theta = \frac{\pi}{4}$ (measured from the positive z axis) and $\phi = 0$ (measured from the xz plane to the projection of the state into the xy plane). The remaining eigenvector (with eigenvalue -1) is simply the state orthogonal to $\left|H_+\right\rangle$, or

$$\left|H_-\right\rangle = -\sin\frac{\pi}{8} \left|0\right\rangle + \cos\frac{\pi}{8} \left|1\right\rangle ,$$

   which geometrically is the state antipodal (i.e. diametrically opposed) to $\left|H_+\right\rangle$. Both of these are identical to the eigenstates previously given.

(e) Ignoring the global phase factor, and noting that $H^2 = \mathbb{1}$, regardless of what is the axis of rotation we know that $\theta = \pm\pi$, since two applications of the rotation gives the identity. From the solution above, we know that $\hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$, and thus $H = R_{\frac{1}{\sqrt{2}}(1,0,1)}(\pm\pi)$ up to a global phase factor.

(f) (i) Tracing out Bob's qubit (the second qubit) we have:

$$\begin{aligned}\mathbf{tr}_2\rho &= \frac{1}{2}\mathbf{tr}_2\left(|01\rangle\langle 01| - |01\rangle\langle 10| + |10\rangle\langle 10| - |10\rangle\langle 01|\right)\\ &= \frac{1}{2}\left(|0\rangle\langle 0| - 0 + |1\rangle\langle 1| - 0\right)\\ &= \frac{1}{2}\mathbb{1}.\end{aligned}$$

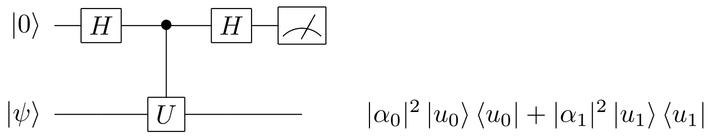
(ii) First Bob applies X to his qubit, then he applies Z. This yields

$$\begin{aligned}&(\mathbb{1}\otimes ZX)\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)\\ =\ &(\mathbb{1}\otimes Z)\frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)\\ =\ &\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).\end{aligned}$$

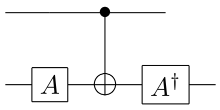
2. (a) In the assignment 2 solutions a rigorous way to obtain a Schmidt decomposition is shown. Using the same technique one should get that the decomposition is:

$$\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)\otimes\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

(b) The circuit is the following



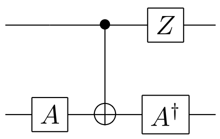
(c) It is immediate that we can implement a controlled $-A$ using the following circuit



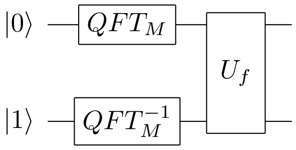
Three marks are given for this.

However, we wish to implement a controlled A. The problem is that we are adding a phase of -1 to the final state when the control qubit is in the state $|1\rangle$. We wish to remove this phase. In other words we wish to multiply the phase of the state by -1 if and only if the control register is in the state $|1\rangle$. It is now evident that we wish to apply a Z gate to the first qubit. This insight is worth three marks.

The final circuit is the following



3. (a) The circuit looks like this:



First note that

$$\text{QFT}\,|1\rangle = \frac{1}{\sqrt{m}}\sum_{k=0}^{m-1} e^{-2\pi ik/m}\,|k\rangle = |\psi_0\rangle$$

Now for any given $x \in Z_m$ we have that:

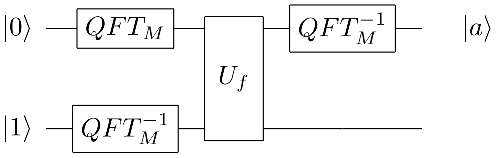
$$\begin{aligned}
U_f\,|x\rangle\,|\psi_0\rangle &= |x\rangle \otimes \left(\frac{1}{\sqrt{m}}\sum_{k=0}^{m-1} e^{-2\pi ik/m}\,|k+f(x)\rangle\right)\\
&= |x\rangle \otimes \left(\frac{1}{\sqrt{m}}\sum_{l=0}^{m-1} e^{-2\pi i(l-f(x)/m}\,|l\rangle\right)\\
&= e^{2\pi if(x)/m}\,|x\rangle \otimes \left(\frac{1}{\sqrt{m}}\sum_{l=0}^{m-1} e^{2\pi il/m}\,|l\rangle\right)\\
&= e^{2\pi if(x)/m}\,|x\rangle \otimes |\psi_0\rangle
\end{aligned}$$

Here we have made the substitution $l = k + f(x)$.
We see that $|\psi_0\rangle$ is an eigenvector of the operator $y \to y + f(x)$ with eigenvalue $f(x)$. Hence, if we put the the first register in an equal superposition of all states $|x\rangle, x \in Z_m$, before applying U_f we get the state $|\psi\rangle \otimes |\psi_0\rangle$.

(b) First note that

$$\begin{aligned}
|\psi\rangle &= \sum_{x=0}^{m-1}\left(e^{2\pi i/m}\right)^{(ax+b)}|x\rangle &(1)\\
&= \left(e^{2\pi i/m}\right)^{b}\sum_{x=0}^{m-1}\left(e^{2\pi i/m}\right)^{(ax)}|x\rangle &(2)\\
&= \sum_{x=0}^{m-1}\left(e^{2\pi iax/m}\right)|x\rangle &(3)
\end{aligned}$$

Where the last equality makes us of the fact the global phases don't matter in quantum states. Applying an inverse QFT on this last state will now give us the state $|a\rangle$. The final circuit to compute a looks like this.



4. (a) The simplest approach is to expand the exponential as a power series

$$U = \exp(-i\alpha X \otimes Z) = \mathbb{1} \otimes \mathbb{1} + \frac{(-i\alpha X \otimes Z)}{1!} + \frac{(-i\alpha X \otimes Z)^2}{2!} + \cdots$$

Note that since $X^2 = Z^2 = \mathbb{1}$, $(X \otimes Z)^{2n} = \mathbb{1}$ and $(X \otimes Z)^{2n+1} = X \otimes Z$ for any integer n, so we get

$$U = \mathbb{1} \otimes \mathbb{1} \left(1 + \frac{(-i\alpha)^2}{2!} + \frac{(-i\alpha)^4}{4!} + \cdots \right) + X \otimes Z \left(\frac{(-i\alpha)}{1!} + \frac{(-i\alpha)^3}{3!} + \cdots \right),$$

which yields

$$U = \cos(\alpha)\mathbb{1} \otimes \mathbb{1} - i\sin(\alpha) X \otimes Z.$$

An alternative (but a lot longer) approach is to note that for any Hermition operator O with eigenvectors $|o_j\rangle$ and corresponding eigenvalues o_j, the unitary operator $e^{i\alpha O}$ has the same eigenvectors, but has corresponding eignevalues $e^{i\alpha o_j}$.
The operator given here, $X \otimes Z$, has eigenvector/eigenvalue pairs

$$\begin{array}{ccc} |+\rangle\,|0\rangle & , & 1 \\ |+\rangle\,|1\rangle & , & -1 \\ |-\rangle\,|0\rangle & , & -1 \\ |-\rangle\,|1\rangle & , & 1. \end{array}$$

We can therefore say that $\exp(-i\alpha X \otimes Z)$ has eigenvector/eigenvalue pairs

$$\begin{array}{ccc} |+\rangle\,|0\rangle & , & \exp(-i\alpha) \\ |+\rangle\,|1\rangle & , & \exp(i\alpha) \\ |-\rangle\,|0\rangle & , & \exp(i\alpha) \\ |-\rangle\,|1\rangle & , & \exp(-i\alpha). \end{array}$$

Writing $\exp(-i\alpha X \otimes Z)$ in its diagonal representation, we have

$$\begin{aligned} \exp(-i\alpha X \otimes Z) \quad &= \quad \exp(-i\alpha)\,|+\rangle\,|0\rangle\,\langle +|\,\langle 0| + \exp(i\alpha)\,|+\rangle\,|1\rangle\,\langle +|\,\langle 1| + \\ &\quad\quad \exp(i\alpha)\,|-\rangle\,|0\rangle\,\langle -|\,\langle 0| + \exp(-i\alpha)\,|+\rangle\,|1\rangle\,\langle +|\,\langle 1|\,. \end{aligned}$$

Noting that

$$\begin{aligned} |0\rangle\,\langle 0| \quad &= \quad \frac{1}{2}(\mathbb{1} + Z) \\ |1\rangle\,\langle 1| \quad &= \quad \frac{1}{2}(\mathbb{1} - Z) \\ |+\rangle\,\langle +| \quad &= \quad \frac{1}{2}(\mathbb{1} + X) \\ |-\rangle\,\langle -| \quad &= \quad \frac{1}{2}(\mathbb{1} - X), \end{aligned}$$

we rewrite the diagonal representation as

$$\begin{aligned} \exp(-i\alpha X \otimes Z) \quad &= \quad \exp(-i\alpha)\frac{1}{4}(\mathbb{1} + X) \otimes (\mathbb{1} + Z) + \exp(i\alpha)\frac{1}{4}(\mathbb{1} + X) \otimes (\mathbb{1} - Z) + \\ &\quad\quad \exp(i\alpha)\frac{1}{4}(\mathbb{1} - X) \otimes (\mathbb{1} + Z) + \exp(-i\alpha)\frac{1}{4}(\mathbb{1} - X) \otimes (\mathbb{1} - Z). \end{aligned}$$

to obtain, after simplification,

$$U = \exp(-i\alpha X \otimes Z) = \cos(\alpha)\mathbb{1} \otimes \mathbb{1} - i\sin(\alpha) X \otimes Z.$$

(b) (i) The initial state is

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |1\rangle = \alpha\,|01\rangle + \beta\,|11\rangle\,.$$

Applying the unitary U, we have

$$\alpha U\,|01\rangle + \beta U\,|11\rangle = \frac{\alpha}{\sqrt{2}}(|01\rangle - |10\rangle) + \frac{\beta}{\sqrt{2}}(|11\rangle - |00\rangle),$$

as the final state of the joint system.

(ii) If we take $\alpha = 1$, then the state after applying U is

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

and tracing out the second subsystem simply yields $\frac{1}{2}\mathbb{1}$ since the joint state is a maximally entangled Bell state.

(iii) We can calculate the result for any α and β, and then simply choose these parameters to give the desired result. In particular, we have

$$\mathbf{tr}_2\rho = \left(\frac{|\alpha|^2}{2} + \frac{|\beta|^2}{2}\right)(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|) + \left(\frac{\alpha\beta^*}{2} + \frac{\alpha^*\beta}{2}\right)(|0\rangle\,\langle 1| + |1\rangle\,\langle 0|)\,,$$

where ρ is the output of the U operation. For the original state, we have

$$|\psi\rangle\,\langle\psi| = |\alpha|^2\,|0\rangle\,\langle 0| + |\beta|^2\,|1\rangle\,\langle 1| + \alpha\beta^*\,|0\rangle\,\langle 1| + \alpha^*\beta\,|1\rangle\,\langle 0|\,,$$

so that in order to obtain $|\psi\rangle\,\langle\psi| = \mathbf{tr}_2\rho$, it is clear that we can set $\alpha = \beta = \frac{1}{\sqrt{2}}$. Moreover, if we set $\alpha = -\beta = \frac{1}{\sqrt{2}}$, the same result holds!

Another alternative is to realize that the only operators applied to the first qubit are $\mathbb{1}$ and X, so that we can choose an common eignestate of both these operators (since $\mathbb{1}$ commutes with everything, this is possible). If we choose $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, again we see that both of these states are invariant under the operation described – note that the same cannot be said about superpositions of these two states.

(c) Yes, it is possible to distinguish between $F(|000\rangle\,\langle 000|)$ and $F(|111\rangle\,\langle 111|)$. This should be clear from the fact that F maps $|000\rangle\,\langle 000|$ to a mixture of states which are orthogonal to the mixture of states which $|111\rangle\,\langle 111|$ is mapped to. Formally, this can be checked by computing the fidelity between the results of applying F to the two different input states

$$\begin{aligned} & \mathbf{tr}\left[(F(|000\rangle\,\langle 000|))^\dagger\, F(|111\rangle\,\langle 111|)\right] \\ = \; & \mathbf{tr}\Big[\frac{1}{16}(|000\rangle\,\langle 000| + |100\rangle\,\langle 100| + |010\rangle\,\langle 010| + |001\rangle\,\langle 001|) \\ & (|111\rangle\,\langle 111| + |011\rangle\,\langle 011| + |101\rangle\,\langle 101| + |110\rangle\,\langle 110|)\Big] \\ = \; & \mathbf{tr}0 = 0, \end{aligned}$$

which means that $|000\rangle$ and $|111\rangle$ can be perfectly disntinguished even after F is applied.

One may think of F as a noisy channel, and $|000\rangle$ and $|111\rangle$ as encoded $|0\rangle$ and encoded $|1\rangle$. This encoding would allow for perfect transmission of quantum data through the noisy channel F as long as error correction is coherently applied at the output of the channel.

In fact, one may check the the encoding $|0\rangle \rightarrow |000\rangle\,, |1\rangle \rightarrow |111\rangle$ satisfies the Knill-Laflamme conditions for an error correction code resilient against the errors A_i.

5. (a) First, we find the eigenstates of ρ, with corresponding eigenvalues. We have $|+\rangle$ with eigenvalue $\frac{2}{3}$ and $|-\rangle$ with eigenvalue $\frac{1}{3}$. This gives us a spectral decomposition of

$$\frac{2}{3} |+\rangle \langle +| + \frac{1}{3} |-\rangle \langle -| .$$

We want a two-system state that will yield ρ upon measurement of the second state. A suitable one is:

$$\sqrt{\frac{2}{3}} |+\rangle |0\rangle + \frac{1}{\sqrt{3}} |-\rangle |1\rangle .$$

(b) (i) Since $HXH = Z$ and $H^2 = \mathbb{1}$, we have $c - Z = (\mathbb{1} \otimes H)CNOT(\mathbb{1} \otimes H)$.

(ii) $U_f = Z \otimes Z$ gives the desired result (notice that $(-1)^{x_1 \oplus x_2} = (-1)^{x_1 + x_2}$).

(c) (i) We have $\left|\frac{a_0}{2}\right|^2 + \left|\frac{a_1}{2}\right|^2 = \frac{1}{4}$.

(ii) The normalized states are $|\psi_1\rangle = \alpha_0 |01\rangle + \alpha_1 |10\rangle$ and $|\psi_0\rangle = \beta_0 |00\rangle + \beta_1 |11\rangle$. We see that $A|00\rangle = \frac{1}{2} |\psi_1\rangle + \frac{\sqrt{3}}{2} |\psi_0\rangle$, so $\theta = \frac{\pi}{6}$.

(d) This is an example of a quantum search. Notice that with $\frac{\pi}{2} = 3\theta$, we need only one Grover iterate, starting with the state $A|00\rangle$. We have $U_f : |\psi_j\rangle \mapsto (-1)^j |\psi_j\rangle$ from part (b)(ii), so all we need now is an implementation of $AU_{00}A^\dagger$ (where $U_{00}|00\rangle = -|00\rangle$ and $U_{00}|x\rangle = |x\rangle$ for $x \neq 00$) or, equivalently, $2|\psi\rangle\langle\psi| - \mathbb{1}$.

Notice that $2|\psi\rangle\langle\psi| - \mathbb{1} = 2A|00\rangle\langle 00| A^\dagger - \mathbb{1} = 2A|00\rangle\langle 00| A^\dagger - AA^\dagger = -A(\mathbb{1} - 2|00\rangle\langle 00|)A^\dagger$.

The operator $U_{00} = \mathbb{1} - 2|00\rangle\langle 00|$ would be like a controlled Z, except the phase shift would occur on the input $|00\rangle$. We can implement this as $(X \otimes X)(c - Z)(X \otimes X)$, implementing $c - Z$ as in part (b)(i). (Alternatively, we could have used $U_{00} = -(Z \otimes Z)(c - Z) = -(c - Z)(Z \otimes Z)$ instead.)

Putting these together, ignoring global phases and expanding $c - Z$, we have

$$|\psi_1\rangle = A(X \otimes X)(\mathbb{1} \otimes H)CNOT(\mathbb{1} \otimes H)(X \otimes X)A^\dagger U_f A|00\rangle .$$