

Interpretation of Quantum Theory, Phys 490/773
Problem Sheet 1
Due: February 1st

1. *Uncertainty Principles in Quantum Mechanics.*

- a) Prove Robertson's uncertainty principle,

$$\Delta A \Delta B \geq \langle (1/2) | [A, B] \rangle$$

for Hermitian operators A and B , where $(\Delta A)^2 \equiv \langle A^2 \rangle - \langle A \rangle^2$ and $(\Delta B)^2 \equiv \langle B^2 \rangle - \langle B \rangle^2$. You may follow the original proof, as published by Robertson in Phys. Rev. 34, 163 (1929).

- b) Explain the operational significance of the quantities ΔA and ΔB .
c) Explain the operational significance of Robertson's uncertainty principle.
d) Explain the operational significance of the uncertainty relation $\delta x \delta p > \hbar$ that may be inferred from the Heisenberg microscope analysis.

2. *Pure vs Mixed States and their Matrix Representations.*

Consider the pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, which is a coherent superposition of the basis states $\{|0\rangle, |1\rangle\}$ for a two-level system.

- a) Express the state operator $\hat{\rho}_p = |\psi\rangle\langle\psi|$ as a matrix in the $\{|0\rangle, |1\rangle\}$ basis.
b) Express the state operator $\hat{\rho}_m = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$ as a matrix in the same basis.
c) Consider the new basis $\{|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\}$. Express both $\hat{\rho}_p$ and $\hat{\rho}_m$ as matrices in this new basis.
d) Determine a condition on α, β so that $\hat{\rho}_m$ is a diagonal matrix in every basis.
e) What is the relationship between the coherence/incoherence of a superposition over two basis vectors and the off-diagonal matrix elements of the state operator expressed in the basis defined by these vectors?

3. *Schmidt Decomposition and Degree of Purity.*

The Schmidt decomposition is defined as follows: for all $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there are bases $\{|\chi_{A,i}\rangle\}$ and $\{|\chi_{B,i}\rangle\}$ for \mathcal{H}_A and \mathcal{H}_B respectively such that

$$|\psi\rangle = \sum_i \alpha_i |\chi_{A,i}\rangle \otimes |\chi_{B,i}\rangle \quad (1)$$

where $\alpha_i \geq 0$, $\sum_i \alpha_i^2 = 1$. Note that a state $|\psi\rangle$ is 'entangled' if and only if the number of non-zero α_i is greater than 1.

Use the Schmidt decomposition to show that

$$\text{Tr}(\hat{\rho}_A^2) = \text{Tr}(\hat{\rho}_B^2) \quad \text{if} \quad |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B,$$

where the reduced density operators are defined by partial traces: $\rho_A = \text{Tr}_B(\rho)$ and $\rho_B = \text{Tr}_A(\rho)$.

4. *Coherent and incoherent superpositions.*

If we have two orthonormal states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, we can 'add' them to give another normalized state ρ in at least two different ways. We can add them incoherently,

$$\rho_{\text{incoh}} = p_1 |\Psi_1\rangle\langle\Psi_1| + p_2 |\Psi_2\rangle\langle\Psi_2|,$$

with probabilities p_1 and p_2 , or we can add them coherently,

$$\Psi = \sqrt{p_1}|\Psi_1\rangle + e^{i\theta}\sqrt{p_2}|\Psi_2\rangle$$

defining the state operator,

$$\rho_{\text{coh}} = |\Psi\rangle\langle\Psi|,$$

where $\sqrt{p_1}$ and $\sqrt{p_2}$ are amplitudes and $e^{i\theta}$ defines a relative phase.

- Calculate $\text{Tr}(\rho_{\text{incoh}}^2)$ and $\text{Tr}(\rho_{\text{coh}}^2)$ and explain how this quantity provides a quantitative measure of the difference between these two states.
- For a Hilbert space of dimension 2 with basis states $\{|0\rangle, |1\rangle\}$, and with $p_0 = 1/3$ and $p_2 = 2/3$, find an observable (i.e., a 2×2 Hermitian matrix defined in this basis) whose expectation value can distinguish between a coherent and incoherent superposition and find an observable whose expectation value can not distinguish between coherent and incoherent superpositions.
- Show that the ('Bell') state

$$|\Psi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} = (|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b)/\sqrt{2}$$

(defined for two two-level systems labeled by 'a' and 'b') can not be written as a state operator of the form,

$$\rho = \sum_i p_i \rho_a^i \otimes \rho_b^i,$$

where $p_i > 0$ and therefore explain why you can deduce that $|\Psi_+\rangle$ is entangled.

- How many independent parameters are required to describe the most general unitary transformation of a two state system? How many independent parameters describe the most general completely positive map?
- Consider an initial factorable state of two spin-1/2 particles with basis states $|1\rangle$ and $|-1\rangle$ defined as the eigenstates of the Pauli operator σ_z . Let the initial state $|1\rangle|1\rangle$ evolve under a unitary operator U , such that $|\Psi\rangle = U|1\rangle|1\rangle$, where,

$$U = (U^{(A)}(\varphi_A) \otimes U^{(B)}(\varphi_B))U_{AB}(J)(U^{(A)}(\theta_A) \otimes U^{(B)}(\theta_B))$$

where

$$U^{(A,B)}(\varphi_{A,B}) = e^{-i\varphi_{A,B}\sigma_x^{(A,B)}}$$

and,

$$U_{AB}(J) = e^{-i\frac{J}{4}\sigma_z^{(A)} \otimes \sigma_z^{(B)}}$$

and let $\theta_A = \theta_B = \pi/4$, so that $U^{(A)}(\theta_A) = \sigma_x^{(A)}$ and $U^{(B)}(\theta_B) = \sigma_x^{(B)}$.

- Calculate the the final state reduced density operator for system A defined as $\rho_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$.
- Find the value of J such that $\text{Tr}(\rho_A^2) = 1/2$.
- Does $\text{Tr}(\rho_A^2)$ depend on φ_A or φ_B ? Why?